On realizations and representations of simple Lie algebra $\mathfrak{sl}(2,\mathbb{C})$

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Abstract. We consider realizations and matrix representations of the simple three-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ over the field of complex numbers. Using a direct method based on solving systems of first-order partial differential equations derived from the algebra's commutation relations, we find all faithful realizations up to equivalence. The obtained exhaustive list of realizations is compared with the one over the real field. The explicit correspondence between two realizations involving nonlinear coefficients in two variables and two matrix representations is established. We also present an example of a realization that cannot be linearized. Some possible applications of realizations and further investigations are pointed out.

Keywords: simple Lie algebras; realizations; vector fields; representations.

1 Introduction

Although Sophus Lie himself initiated the construction of realizations, the problem remains unsolved for many important cases. The description of Lie algebra representations through vector fields is of significant interest and has broad applicability. This includes, in particular, the integration of ordinary differential equations, the group classification of partial differential equations, and the classification of gravity fields in general form under motion groups or groups of conformal transformations, among others (see additional applications in [5]).

Present paper is organized as follows. First, we provide basic definitions and notations in Section 2, then, in Section 3, we list all inequivalent realizations of the complex three-dimensional special linear algebra and discuss connections between realizations and representations of $\mathfrak{sl}(2,\mathbb{C})$.

2 Preliminaries

Let V be an n-dimensional complex vector space together with a bilinear antisymmetric multiplication operation $[\cdot,\ \cdot]\colon V\times V\to V$ satisfying the Jacobi identity

$$\forall x,y,z\in\mathfrak{g}\quad [x,y]=-[y,x],\qquad [[x,y],z]+[[y,z],x]+[[z,x],y]=0.$$

The operation $[\cdot,\cdot]$ is called a *Lie product* (a commutator) and an algebra $\mathfrak{g}=(V,[\cdot,\ \cdot])$ is called an n-dimensional *Lie algebra* over $\mathbb{C}$. Let $\langle e_1,\dots,e_n\rangle$ be a basis of V, then the Lie algebra $\mathfrak{g}$ is defined by its commutation relations $[e_i,e_j]=\sum_{k=1}^n C_{ij}^k e_k$, with the structure constants tensor $C_{ij}^k\in\mathbb{C}$, $i,j,k=1,\dots,n$. Below, we omit the sum symbol and use the summation convention for repeated indices. We also assume that the indices i, j, k, $\tilde{i}$, $\tilde{j}$ and $\tilde{k}$ run from 1 to n.

General linear group acts on $\mathfrak{g}$ as a basis change. Namely, consider a matrix $A=A_j^i\in\mathrm{GL}_n(V)$ and let $B=A^{-1}$ be the inverse of A, then the corresponding basis change $\tilde{e}_j=A_j^i e_i$ in the vector space V leads to the following transformation of the Lie products and structure constants:

$$[\tilde{e}_{\tilde{i}}, \tilde{e}_{\tilde{j}}] = [A^i_{\tilde{i}} e_i, A^j_{\tilde{j}} e_j] = A^i_{\tilde{i}} A^j_{\tilde{j}} C^k_{ij} e_k = A^i_{\tilde{i}} A^j_{\tilde{j}} B^{\tilde{k}}_k C^k_{ij} \tilde{e}_{\tilde{k}} = \tilde{C}^{\tilde{k}}_{\tilde{i}\tilde{j}} e_{\tilde{k}}.$$

Denote the group of all the automorphisms of the Lie algebra $\mathfrak{g}$ (which preserve the structure constants) by $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}_n(V)$ and the group of its inner automorphisms by $\mathrm{Inn}(\mathfrak{g}) \subseteq \mathrm{Aut}(\mathfrak{g})$.

The main object of this paper is the unique simple complex three-dimensional Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, which is defined as the algebra of 2×2 traceless complex matrices. The standard choice of basis for this algebra is

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

which satisfy the commutation relations

$$[e, f] = h, \qquad [h, e] = 2e, \qquad [f, h] = 2f. \tag{2}$$

There are other bases of $\mathfrak{sl}(2,\mathbb{C})$, for example, the Hermitian basis is given by the Pauli matrices. For our purposes, it is convenient to choose the basis $e_1 = e$, $e_2 = -\frac{1}{2}h$, $e_3 = -f$ with the commutation relations

$$[e_1, e_2] = e_1, \qquad [e_2, e_3] = e_3, \qquad [e_1, e_3] = 2e_2. \tag{3}$$

In this work, we adopt the main definitions and the method for the construction of realizations developed in [5]. Let $M \subset \mathbb{C}^m$ be an m-dimensional domain, $m \in \mathbb{N}$, and $\mathrm{Vect}(M)$ denote vector fields on M with analytical coefficients. Roughly speaking, we consider the Lie algebra spanned by homogeneous first-order differential operators with the coefficients ξ^α that are analytical functions on M,

$$\xi^1(x_1, x_2, \dots, x_m)\frac{\partial}{\partial x_1} + \xi^2(x_1, x_2, \dots, x_m)\frac{\partial}{\partial x_2} + \dots + \xi^m(x_1, x_2, \dots, x_m)\frac{\partial}{\partial x_m}. \tag{4}$$

Hereafter, indices α and β run from 1 to m, $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$ and x is a short notation for the set of all coordinates $x_1, x_2, \dots, x_m$.

A Lie multiplication of operators of the form (4) is given by the commutator

$$[\xi^\alpha(x)\partial_\alpha, \eta^\beta(x)\partial_\beta] = \xi^\alpha(x)\cdot(\eta^\beta(x))'_\alpha \partial_\beta - \eta^\beta(x)(\xi^\alpha(x))'_\beta \partial_\alpha.$$

Definition 1. A homomorphism $R\colon \mathfrak{g} \to \mathrm{Vect}(M)$ is called a *realization* of a Lie algebra $\mathfrak{g}$ in vector fields on a domain M. Two realizations $R_1\colon \mathfrak{g} \to \mathrm{Vect}(M_1)$ and $R_2\colon \mathfrak{g} \to \mathrm{Vect}(M_2)$ are called *equivalent* if there exist an automorphism transformation $A \in \mathrm{Aut}(\mathfrak{g})$ and an invertible function $f\colon M_1 \to M_2$ with the induced isomorphism $f_*\colon \mathrm{Vect}(M_2) \to \mathrm{Vect}(M_1)$ such that for all $g \in \mathfrak{g}$ we have $R_1(g) = f_* R_2(Ag)$. If $\mathrm{Ker}(R) = \{0\}$, then a realization R is called *faithful*.

Note, that we work only locally and non-local classification of realizations can contain more cases, that are locally equivalent. At the same time, we can apply the obtained operators not only locally, but everywhere where they are defined.

To construct a realization R of a Lie algebra $\mathfrak{g}$ by the direct method, we start from given commutation relations and assume that the basis elements of $\mathfrak{g}$ are of the general form (4): $R(e_i) = \xi^\alpha_i(x)\partial_\alpha$. Substituting basis elements into the commutation relations, we obtain a system of first-order partial differential equations for the coefficients $\xi^\alpha_i(x)$. By solving such systems and investigating the equivalence, we obtain the desired classification.

This method is significantly simplified in the case when one of the basis operators is reduced to the shift operator ∂_1 using invertible transformations of the domain M.

We also can construct realizations step by step, starting from the one-dimensional algebra to two-dimensional, and so on. In this case, the key role is played by the megaideals of co-dimension one (a vector subspaces of $\mathfrak{g}$ invariant under $\mathrm{Aut}(\mathfrak{g})$). Such the $(n-1)$-dimensional invariant subspaces do exist for all solvable Lie algebras, but not for the unsolvable ones. For simple Lie algebras, not only megaideals of co-dimension one do not exist, but for some of them even $(n-1)$-dimensional subalgebras do not exist, an example of such an algebra is $\mathfrak{so}(3)$ — the real three-dimensional algebra of the rotation group. This significantly complicates the construction of all realizations of unsolvable Lie algebras and the task of proving their non-equivalence.

Another important concept that simplifies the study of realizations is the realization rank. Let us fix a point $x \in M$ and consider the $n \times m$ matrix formed by the coefficients of basis operators at this point

$$\xi(x) = \begin{pmatrix} \xi_1^1(x) & \xi_2^1(x) & \dots & \xi_m^1(x) \\ \xi_1^2(x) & \xi_2^2(x) & \dots & \xi_m^2(x) \\ \vdots & \vdots & \ddots & \vdots \\ \xi_1^n(x) & \xi_2^n(x) & \dots & \xi_m^n(x) \end{pmatrix}.$$

We will call the rank of the matrix $\xi(x)$ at the point x *the rank of realization R at point x* and denote it by $\mathrm{rank}(R_x)$. Note that the rank value satisfies the inequality $0 \leq \mathrm{rank}(R_x) \leq n$, where the upper bound n corresponds to the number of basis elements of $\mathfrak{g}$. The notion of *realization rank* can also be introduced by considering the maximum rank attained across the entire domain M, $\mathrm{rank}\, R = \max_{x \in M}\{\mathrm{rank}(R_x)\}$. The rank of realization is an important invariant quantity, since it does not change under the action of the group of automorphisms and under the action of diffeomorphisms of the manifold M.

3 Realizations and representations of $\mathfrak{sl}(2,\mathbb{C})$

Using the direct method described above, we classified all inequivalent realizations of the complex Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ starting from the commutation relations for the basis e_1, e_2, e_3.

Theorem 2. *There are only four inequivalent faithful realizations of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ with the commutation relations (3), which are exhausted by the list:*

(i) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2 + x_2\partial_3$;

(ii) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ \big(x_1^2 + x_2^2\big)\partial_1 + 2x_1x_2\partial_2$;

(iii) $\partial_1,\ x_1\partial_1 + x_2\partial_2,\ x_1^2\partial_1 + 2x_1x_2\partial_2$;

(iv) $\partial_1,\ x_1\partial_1,\ x_1^2\partial_1$.

It was proven in [4] that all inequivalent realizations of the real Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ coincide with the list (i)–(iv) up to one additional realization

$$R_3(e_1) = \partial_1, \qquad R_3(e_2) = x_1\partial_1 + x_2\partial_2, \qquad R_3(e_3) = \big(x_1^2 - x_2^2\big)\partial_1 + 2x_1x_2\partial_2.$$

If we apply a nondegenerate coordinate change $\tilde{x}_1 = x_1$, $\tilde{x}_2 = \mathrm{i}x_2$ to the realization R_3, where i is an imaginary unit, then it takes the form

$$\tilde{R}_3(e_1) = \partial_{\tilde{x}_1}, \qquad \tilde{R}_3(e_2) = (x_1\cdot 1 + x_2\cdot 0)\partial_{\tilde{x}_1} + (x_1\cdot 0 + x_2\cdot \mathrm{i})\partial_{\tilde{x}_2} = \tilde{x}_1\partial_{\tilde{x}_1} + \tilde{x}_2\partial_{\tilde{x}_2},$$
$$\tilde{R}_3(e_3) = \big(x_1^2 - x_2^2\big)\partial_{\tilde{x}_1} + 2x_1\mathrm{i}x_2\partial_{\tilde{x}_2} = \big(\tilde{x}_1^2 + \tilde{x}_2^2\big)\partial_{\tilde{x}_1} + 2\tilde{x}_1\tilde{x}_2\partial_{\tilde{x}_2}.$$

Now it is clear that over the complex field the realization R_3 is equivalent to the case (ii) from the theorem list.

Let us compare the ranks of the realizations in the theorem list. Realization (i) has rank three, and realization (iv) has rank one, so they cannot be equivalent to any of the other realizations. The only pair of realizations that may be equivalent are realizations (ii) and (iii).

Our next task is to consider matrix representations of $\mathfrak{sl}(2,\mathbb{C})$ and examine how they are related to inequivalent realizations listed in Theorem 2.

Let us define how an abstract Lie algebra $\mathfrak{g}$ can be viewed as a subalgebra of endomorphism algebra of an n-dimensional complex vector space V (this vector space may not coincide with the vector space on which the Lie algebra is defined).

Definition 3. A *representation* of a Lie algebra $\mathfrak{g}$ is a Lie algebra homomorphism $\varphi\colon \mathfrak{g} \to \mathfrak{gl}(V)$.

For each Lie algebra, there are always two representations: trivial and adjoint. A trivial representation is one in which each element of the algebra is mapped to the zero element of the vector space, and such a representation has no essential applications.

The adjoint representation for $V = \mathfrak{g}$ is given by the map

$$\mathrm{ad}\colon\ \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \qquad \mathrm{ad}_x y = [x,y] \quad \text{for all } x,y \in \mathfrak{g}.$$

Consider the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$ with the basis commutation relations (3),

$$\mathrm{ad}_{e_1} = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}, \quad \mathrm{ad}_{e_2} = \begin{pmatrix} -1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \quad \mathrm{ad}_{e_3} = \begin{pmatrix} 0 & 0 & 0\\ -2 & 0 & 0\\ 0 & -1 & 0\end{pmatrix}. \tag{5}$$

Another type of representation that always exists for Lie algebras defined over a vector space of finite-dimensional matrices is the natural representation, i.e., the embedding of a Lie algebra into the corresponding matrix algebra. For the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$, the natural representation of basis elements can be taken in the form (1).

In general case, all irreducible representations of $\mathfrak{sl}(2,\mathbb{C})$ can be constructed as weight representations or as homomorphisms on the complex vector spaces of homogeneous polynomials of two variables, for more details see, e.g., [2]. Here we present the general result in the form of $(d+1)\times(d+1)$ matrices ($d \in \mathbb{N}$) for the homomorphism φ of the basis elements e, f, h with the commutation relations (2),

$$\begin{gathered}\varphi(e) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & d\\ 0 & 0 & 0 & \cdots & 0\end{pmatrix}, \qquad \varphi(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0\\ d & 0 & \cdots & 0 & 0\\ 0 & d-1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{pmatrix},\\ \varphi(h) = \begin{pmatrix} d & 0 & \cdots & 0 & 0\\ 0 & d-2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & -d+2 & 0\\ 0 & 0 & \cdots & 0 & -d\end{pmatrix}.\end{gathered} \tag{6}$$

Analyzing the general form of the representations, we can conclude that the natural representation corresponds to the case $d = 1$ and the adjoint one corresponds to the case $d = 2$.

Our next task is to establish a correspondence between the representations and the realizations of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. To begin with, let us note that it is possible to establish a one-to-one correspondence between realizations with linear coefficients and matrix representations. Such a correspondence follows from the fact that the subspace of vector fields with linear coefficients is actually a Lie subalgebra and is isomorphic to the general linear Lie algebra, see, e.g., [1]. For practical applications, it is convenient to express this statement in a non-invariant form, employing the structure constant tensor and the representation matrices.

Lemma 4. *Let $\mathfrak{g}$ be a Lie algebra given by commutation relations $[e_i,e_j]=C_{ij}^k e_k$. If this algebra has a representation φ with the corresponding $m\times m$ matrices $\varphi(e_i)=\Phi_i$, where the matrix elements are $(\Phi_i)_\alpha^\beta$, then the set of first-order differential operators with the linear coefficients*

$$\Xi_i=\sum_{\alpha=1}^{m}\left(\sum_{\beta=1}^{m}(\Phi_i)_\alpha^\beta x_\beta\right)\partial_\alpha$$

is a realization $R(e_i)=\Xi_i$ of $\mathfrak{g}$ and vice versa: any realization with linear coefficients generates the matrix representation of the Lie algebra.

3.1 Example of the natural representation

Let us consider the matrix representation (6) for the case $d=1$, it coincides with the natural representation (1). Using Lemma 4, we construct the corresponding realization with linear coefficients

$$R(e)=x_1\partial_2,\qquad R(f)=x_2\partial_1,\qquad R(h)=x_1\partial_1-x_2\partial_2.$$

In order to compare this realization with the list of inequivalent cases presented in Theorem 2, we express it in the e-basis

$$R(e_1)=x_1\partial_2,\qquad R(e_2)=-\tfrac{1}{2}x_1\partial_1+\tfrac{1}{2}x_2\partial_2,\qquad R(e_3)=-x_2\partial_1.$$

To establish which of the realizations from Theorem 2 this realization is equivalent to, we reduce the operator $x_1\partial_2$ to the shift operator $\partial_{\tilde{x}_1}$. To proceed, we perform the change of variables $\tilde{x}_1=\frac{x_1}{x_2}$, $\tilde{x}_2=x_1$, under which the realization takes the form

$$\tilde{R}(e_1)=\partial_{\tilde{x}_1},\qquad \tilde{R}(e_2)=\tilde{x}_1\partial_{\tilde{x}_1}-\tfrac{1}{2}\tilde{x}_2\partial_{\tilde{x}_2},\qquad \tilde{R}(e_3)=\tilde{x}_1^2\partial_{\tilde{x}_1}-\tilde{x}_1\tilde{x}_2\partial_{\tilde{x}_2}.$$

To reduce the realization $\tilde{R}$ to the realization (iii) from Theorem 2, we apply the additional transformation $\tilde{\tilde{x}}_1=\tilde{x}_1$, $\tilde{\tilde{x}}_2=\frac{1}{\tilde{x}_2^2}$ and we get

$$\tilde{\tilde{R}}(e_1)=\partial_{\tilde{\tilde{x}}_1},\qquad \tilde{\tilde{R}}(e_2)=\tilde{\tilde{x}}_1\partial_{\tilde{\tilde{x}}_1}+\tilde{\tilde{x}}_2\partial_{\tilde{\tilde{x}}_2},\qquad \tilde{\tilde{R}}(e_3)=\tilde{\tilde{x}}_1^2\partial_{\tilde{\tilde{x}}_1}+2\tilde{\tilde{x}}_1\tilde{\tilde{x}}_2\partial_{\tilde{\tilde{x}}_2}.$$

Therefore, the realization constructed from the natural representation is equivalent to the nonlinear realization (iii) in Theorem 2. Since the transformations, we constructed are invertible (except for the case $x_2=0$, but this is the case of unfaithful realization), we can construct the inverse transformation to their composition and embed the nonlinear realization into a linear space.

3.2 Example of the adjoint representation

Let us consider the matrix representation for the case $d=2$ which is the adjoint representation (5). Its corresponding realization with linear coefficients has the form

$$R(e_1)=x_1\partial_2+2x_2\partial_3,\qquad R(e_2)=-x_1\partial_1-x_3\partial_3,\qquad R(e_3)=-2x_2\partial_1-x_3\partial_2.$$

To reduce this realization to the case (ii) from Theorem 2, we perform the transformations

$$\tilde{x}_1=\frac{x_2+1}{x_1}-x_2^2+x_1x_3,\qquad \tilde{x}_2=x_1^{-1/2}\big(x_2^2-x_1x_3\big),\qquad \tilde{x}_3=\frac{1}{x_1}$$

and we get

$$\tilde{R}(e_1)=\partial_{\tilde{x}_1},\qquad \tilde{R}(e_2)=\tilde{x}_1\partial_{\tilde{x}_1}+\tilde{x}_2\partial_{\tilde{x}_2},\qquad \tilde{R}(e_3)=\big(\tilde{x}_1^2+\tilde{x}_2^2\big)\partial_{\tilde{x}_1}+2\tilde{x}_1\tilde{x}_2\partial_{\tilde{x}_2}.$$

Therefore, the realization constructed from the adjoint representation is equivalent to the nonlinear realization (ii) in Theorem 2.

3.3 Realizations of the ranks one and three

Irreducible representations (6) for $d \geq 3$ correspond to rank-three realizations and should be equivalent to the case (i) of Theorem 2; however, the transformations that reduce them to this case are quite complicated and cumbersome.

We will show that the realization $e_1 = \partial_1$, $e_2 = x_1\partial_1$, $e_3 = x_1^2\partial_1$ (case (iv) from Theorem 2) cannot be represented as a realization with linear homogeneous coefficients. It is evident that this realization cannot be linearized via any non-degenerate transformation of the basis elements. Therefore, it suffices to prove that no non-degenerate change of coordinates can transform this realization into a linear one. Suppose that there exists a change of coordinates with a non-vanishing Jacobian:

$$\tilde{x}_1 = f_1(x_1), \quad \tilde{x}_2 = f_2(x_1), \quad \dots, \quad \tilde{x}_m = f_m(x_1), \qquad J(f_1(x_1), f_2(x_1), \dots, f_m(x_1)) \neq 0,$$

which transforms the second and third operators into the form

$$\tilde{e}_2 = a_{\alpha\beta} f_\beta \partial_{\tilde{x}_\alpha}, \qquad \tilde{e}_3 = b_{\alpha\beta} f_\beta \partial_{\tilde{x}_\alpha},$$

where $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\alpha, \beta = 1, \dots, m$, are complex constants. Constructing the corresponding system of partial differential equations for these operators and using the condition $\tilde{e}_3 = x_1\tilde{e}_2$, we obtain a system of functional (not differential) equations that is linear and homogeneous in the functions $f_1, \dots, f_m$. This contradicts the assumption that the Jacobian $J(f_1(x_1), f_2(x_1), \dots, f_m(x_1)) \neq 0$. Hence, such a coordinate transformation does not exist.

Most likely, this case corresponds to an infinite-dimensional matrix representation; however, this issue is quite complex [3] and will be the subject of our future investigations.

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Про реалізації та зображення простої алгебри Лі $\mathfrak{sl}(2,\mathbb{C})$

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Анотація. Розглянуто реалізації та матричні зображення простої тривимірної алгебри Лі $\mathfrak{sl}(2,\mathbb{C})$ над полем комплексних чисел. Застосовуючи прямий метод, що полягає у розв'язанні систем диференціальних рівнянь у частинних похідних, отриманих з комутаційних співвідношень алгебри, знайдено усі точні реалізації з точністю до еквівалентності. Встановлено відповідність між отриманим вичерпним переліком та аналогічним списком реалізацій для випадку поля дійсних чисел. Для двох реалізацій рангу два з нелінійними коефіцієнтами представлено явний вигляд перетворень, що зводять їх до матричних зображень. Також у роботі наведено приклад реалізації, що нееквівалентна жодному скінченовимірному матричному зображенню. Вказано можливі застосування і подальший напрямок досліджень.

Ключові слова: прості алгебри Лі; реалізації; векторні поля; зображення